Homework 3

Math 198: Math for Machine Learning

Due Date:
Name:
Student ID:

Instructions for Submission

Please include your name and student ID at the top of your homework submission. You may submit handwritten solutions or typed ones (LaTeX preferred). If you at any point write code to help you solve a problem, please include your code at the end of the homework assignment, and mark which code goes with which problem. Homework is due by start of lecture on the due date; it may be submitted in-person at lecture or by emailing a PDF to both facilitators.

1 Practice with Determinant and Trace

1. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map that sends $(1,0) \mapsto (2,0)$ and $(0,1) \mapsto (3,4)$. What is $\det(T)$?
2. What is $\operatorname{tr}(T)$?
3. Let U be a proper subspace of a vector space V. Let $\mathbf{P}$ be a projection map onto U. What is $\det(\mathbf{P})$?

2 Proofs about Determinant and Trace

Let $\mathbf{A}$ be an arbitrary square matrix.

1. Prove that, if $\mathbf{A}$ is invertible, then $\det(\mathbf{A}^{-1}) = \det(\mathbf{A})^{-1}$.
2. Conclude that if $\det(\mathbf{A}) = 0$, $\mathbf{A}$ is not invertible.
3. Let $\mathbf{B}$ be an invertible matrix. Prove $\operatorname{tr}(\mathbf{A}) = \operatorname{tr}(\mathbf{BAB^{-1}})$.

3 Computing Eigenvalues and Eigenvectors

Let

$$\mathbf{A} = \begin{bmatrix} 4 & 1 & -1 \\ \frac{3}{2} & \frac{7}{2} & -\frac{3}{2} \\ \frac{1}{2} & \frac{-1}{2} & \frac{5}{2} \end{bmatrix}$$

1. Find $p_{\mathbf{A}}(\lambda)$, the characteristic polynomial of $\mathbf{A}$.
2. Using $p_{\mathbf{A}}(\lambda)$, compute the eigenvalues of $\mathbf{A}$.
3. Find the eigenvectors of $\mathbf{A}$.

4 Proofs about Eigenvalues

Let $\mathbf{A} \in \mathbb{R}^{n \times n}$ be a square matrix such that the eigenvectors of $\mathbf{A}$ are a basis for $\mathbb{R}^n$. Additionally, let λ_i, $1 \leq i \leq n$ be the eigenvalues of $\mathbf{A}$.

1. Prove that $\det(\mathbf{A}) = \prod_{i=1}^{n} \lambda_i$.

2. Prove that $\operatorname{tr}(\mathbf{A}) = \sum_{i=1}^{n} \lambda_i$.