Homework 4

Math 198: Math for Machine Learning

Due Date: March 4
Name:
Student ID:

Instructions for Submission

Please include your name and student ID at the top of your homework submission. You may submit handwritten solutions or typed ones (LaTeX preferred). If you at any point write code to help you solve a problem, please include your code at the end of the homework assignment, and mark which code goes with which problem. Homework is due by start of lecture on the due date; it may be submitted in-person at lecture or by emailing a PDF to both facilitators.

1 Projections

Let $\mathbf{P} : V \to V$ be a (not necessarily orthogonal) projection operator, i.e. $\mathbf{P}^2 = \mathbf{P}$.

1. Show that all eigenvalues of $\mathbf{P}$ are either 0 or 1.
2. Show that $\text{tr}(\mathbf{P}) = \text{rank}(\mathbf{P})$.
3. Prove that $\mathbf{P}$ is the identity matrix when restricted to its range. That is, for any vector $\mathbf{v} \in \text{range}(\mathbf{P})$, $\mathbf{P}\mathbf{v} = \mathbf{v}$.
4. Prove that $\mathbf{P}$ is either not invertible or its own inverse.

2 Using the Spectral Theorem

1. Prove that the matrix
$$\mathbf{A} = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$$
is normal.
2. Compute the eigenvalues of $\mathbf{A}$.
3. Compute the eigenvectors of $\mathbf{A}$.
4. Using the results of (a), (b), and (c), combined with your knowledge of the Spectral Theorem, compute the eigendecomposition of $\mathbf{A}$ (in particular, find the orthogonal matrix $\mathbf{Q}$ and diagonal matrix $\mathbf{\Lambda}$ such that $\mathbf{A} = \mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^\top$).
5. Compute $\mathbf{A}^{20}$.

3 PSD Matrices

1. Prove that the two definitions given for positive semi-definite matrices are equivalent. *(Hint: use Rayleigh quotients.)*
2. Prove that, for any matrix $\mathbf{X}$ and any scalar $\lambda > 0$, $\mathbf{X}^\top \mathbf{X} + \lambda \mathbf{I}$ is invertible.