Homework 5

Math 198: Math for Machine Learning

Due Date: March 11
Name:
Student ID:

Instructions for Submission

Please include your name and student ID at the top of your homework submission. You may submit handwritten solutions or typed ones (LaTeX preferred). If you at any point write code to help you solve a problem, please include your code at the end of the homework assignment, and mark which code goes with which problem. Homework is due by start of lecture on the due date; it may be submitted in-person at lecture or by emailing a PDF to both facilitators.

1 Working with Adjoints

Let $\mathbf{A} \in \mathbb{R}^{m \times n}$.

(a) Show that $\ker \mathbf{A}^\top \mathbf{A} = \ker \mathbf{A}$.

(b) Deduce that $\operatorname{rank}(\mathbf{A}^\top \mathbf{A}) = \operatorname{rank}(\mathbf{A})$.

(c) Suppose $\mathbf{A}$ is square. Show that $\mathbf{A}$ and $\mathbf{A}^\top$ have the same eigenvalues.

(d) Deduce that $\operatorname{rank}(\mathbf{A}) = \operatorname{rank}(\mathbf{A}^\top)$. (These proofs can be extended to non-square $\mathbf{A}$ by adding rows/-columns of all zeroes until $\mathbf{A}$ is square.)

2 SVD

(a) Let $\mathbf{A} \in \mathbb{R}^{m \times n}$. Let $\mathbf{A} = \mathbf{U}\Sigma\mathbf{V}^\top$ be its SVD. Find the spectral decompositions of $\mathbf{A}^\top\mathbf{A}$ and $\mathbf{A}\mathbf{A}^\top$ in terms of $\mathbf{U}, \Sigma, \mathbf{V}$.

(b) Prove: If $\mathbf{A} \in \mathbb{R}^{n \times n}$ is PSD, then the spectral decomposition of $\mathbf{A}$ coincides with the SVD of $\mathbf{A}$.

(c) Let $\mathbf{A} \in \mathbb{R}^{m \times n}$, and let $\mathbf{A} = \mathbf{U}\Sigma\mathbf{V}^\top$ be its SVD. If $\mathbf{V} = (\mathbf{v}_1, \ldots, \mathbf{v}_n)$, $\mathbf{U} = (\mathbf{u}_1, \ldots, \mathbf{u}_m)$, and $r = \operatorname{rank}(\mathbf{A})$, then let

$$\mathbf{V}_r = (\mathbf{v}_1, \ldots, \mathbf{v}_r)$$
$$\mathbf{U}_r = (\mathbf{u}_1, \ldots, \mathbf{u}_r).$$

Show that $\mathbf{v}_1, \ldots, \mathbf{v}_r$ "diagonalize" $\mathbf{A}$ in the following way: For $i = 1, \ldots, r$, show that $\mathbf{A}\mathbf{v}_i = \sigma_i \mathbf{u}_i$.

(d) Let

$$\mathbf{A} = \begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{pmatrix}.$$

Compute the SVD of $\mathbf{A}$.

(e) Let

$$\mathbf{A} = \begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{pmatrix}.$$

Find orthonormal bases for the four fundamental subspaces of $\mathbf{A}$.

3 PCA

Let

$$\mathbf{X} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \\ -2 & -2 \\ 0 & 0 \\ 2 & 2 \end{bmatrix}$$

Note that the SVD of $\mathbf{X}$ is

$$\text{SVD}(\mathbf{X}) = \begin{bmatrix} 0 & 0.7 & -0.5 & 0 & 0.5 \\ 0 & -0.7 & -0.5 & 0 & 0.5 \\ -0.7 & 0 & 0.5 & 0 & 0.5 \\ 0 & 0 & 0 & 1 & 0 \\ 0.7 & 0 & 0.5 & 0 & 0.5 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0.7 & 0.7 \\ -0.7 & 0.7 \end{bmatrix}$$

Compute the principal components of $\mathbf{X}$.