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Stochastic Dynamic Pricing: Utilizing Demand Response in an Adaptive Manner

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Abstract— **Dynamic pricing to residential customers has been proposed recently, as extensions of static network utility maximization problems. Those deterministic models do not exploit the refined information as time advances. To address this issue, we formulate a stochastic dynamic pricing framework, in which we show the existence of an optimal price process that incentives the agents to choose the socially optimal decisions. We develop a distributed algorithm and investigate the information structure of the involved stochastic processes via a numerical example, which also illustrates the advantage of stochastic dynamic pricing over deterministic dynamic pricing.**

I. INTRODUCTION

In most retail electricity markets, pricing to residential customers has followed the standard practice of a flat rate structure. Utility companies (or distributors) charge household customers (or users) a fixed price per unit for electricity use regardless of the cost of supply at the time of consumption. As a variant, a tiered rate structure is often adopted that depends on the monthly energy usage. Such inflexible schemes are of no avail in incentivizing users to change their consumption patterns so as to reduce peak load.

The development of smart grid technologies opens up the possibility of improving the efficiency of energy consumption. Dynamic pricing has been proposed recently. The idea is to coordinate demand response to the benefit of the overall system through financial incentives. The common approach of dynamic pricing is based on network utility maximization (NUM), a well-known framework to study network resource allocation problems [1] since the seminal work by Kelly [2]. A typical NUM problem is to maximize the sum of the users' utilities subject to certain linear constraints (e.g., flow and capacity constraints). The convex structure of the problem leads to dual decomposition methods which correspond to resource allocation via pricing. A tutorial on decomposition methods for NUM can be found in [3]. In the context of demand side management, a discrete-time model with a finite horizon is commonly adopted. A class of recent work studied designing time-varying prices that align individual optimality with social optimality [4], [5], [6], [7]. Such multistage models, regarded as dynamic NUM problems, are natural extensions of static NUM problems. Some more specific models include deadline differentiated pricing [8], electric vehicle charging [9], etc. In practice, the so-called time-of-use plan is a basic implementation of this idea, in which rates for periods with different demand levels are pre-established, typically highest during the peak load hours.

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However, all those models are deterministic, and thus the optimal prices, though time-dependent, are also deterministic. In reality, there is uncertainty in the system. The information sets are augmented as time advances. In such a stochastic setting, a deterministic price vector would not induce the socially optimal decisions. Rather, the optimal price process should evolve according to the up-to-date information; so do the optimal decisions. This is the motivation of our proposed model, called stochastic dynamic pricing that further generalizes dynamic NUM problems.

Demand response could also be done via interruptible power service contracts of varying reliability [10]. [11] studied dynamic pricing for a revenue-maximizing retailer. A novel real-time pricing scheme called monotonic marginal pricing is proposed in [12], while price discrimination is allowed. [13] is close to our work in spirit, though in a different context. Beyond proving the existence of an optimal price process, we develop a distributed algorithm and investigate the information structure.

An important consideration in future grid systems is the possibility for customers to utilize storage technologies to hedge against significant price changes. Moreover, a variety of strategies can be utilized to mimic the behavior of storage, such as opportunistic utilization of electric vehicle availability and deferring appliance load [14] or air conditioning consumption [15], [16]. We consider a simple model for storage, in which each user has its own storage with a sufficiently large capacity. Inclusion of capacity limit, charging rate limit and round-trip efficiency is left for future work. The focus of this paper is to develop an adaptive pricing mechanism in response to the presence of uncertainty.

The paper is organized as follows. In Section II, we formulate the user's problem as a stochastic optimal control problem that is of independent interest. In Section III, we formulate the system problem, a stochastic dynamic NUM problem. We apply dual decomposition methods to show the existence of an optimal price process and develop a distributed algorithm. We investigate the information structure via a numerical example, which also illustrates the advantage of stochastic dynamic pricing over deterministic dynamic pricing. Section IV concludes the paper.

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II. User's Problem: Stochastic Optimal Control

Our first task is to understand how an individual user makes the optimal decisions to maximize his own payoff, given an exogenous price process. We formulate a stochastic optimal control problem, in which the user needs to decide how much power to buy and how much to consume (and therefore how much to store) at each time. We consider both the elastic and the inelastic demand cases.

A. Model

The horizon of interest is typically one day that is discretized into T slots, indexed by $t = 0, 1, \ldots, T-1$. At each time t, the user buys $z_t \geq 0$ and consumes $x_t \geq 0$ amount of power. We call z_t the demand and x_t the consumption at time t. The user has a storage with a sufficiently large capacity. Let $y_t \geq 0$ be the storage level at the beginning of time t, with y_0 given. Denote by y_T the storage level at the end of the horizon. Then the storage dynamics is given by

$$y_{t+1} = y_t + z_t - x_t, \ \forall t.$$

Let π_t be the price of power at time t. We consider (y_t, π_t) as the state variables at time t, and (x_t, z_t) as the decision variables. Define $x = \{x_t, \forall t\}$, $y = \{y_t, \forall t\}$, $z = \{z_t, \forall t\}$ and $\pi = \{\pi_t, \forall t\}$. Assume that the price process π is an exogenous Markov process which is independent of the decision variables (x, z). Conditioned on π_t, the probability density function of π_{t+1} is denoted by $f_t(\cdot|\pi_t)$.

B. Inelastic Demand Case

In this case, the user's demand at each time t is a fixed amount $D_t \geq 0$, so that $x_t = D_t$ for all t. Since $y_t, z_t \geq 0$ for all t, the control constraint at time t given the state (y_t, π_t) is $z_t \geq (D_t - y_t)_+$, where we define $x_+ = \max\{x, 0\}$. The user's problem is to minimize the expected cost:

$$\text{minimize } \mathbb{E}[\textstyle\sum_t \pi_t z_t]. \tag{1}$$

We apply the dynamic programming algorithm to obtain the optimal solution to problem (1).

Proposition 1: The optimal policy of problem (1) has a threshold form as shown in (4).

Proof: Define the cost-to-go functions as

$$J_T(y_T, \pi_T) = 0, \tag{2a}$$

$$J_t(y_t, \pi_t) = \min_{z_t \geq (D_t - y_t)_+} Q_t(y_t, \pi_t, z_t), \tag{2b}$$

where

$$Q_t(y_t, \pi_t, z_t) = \pi_t z_t + \mathbb{E}[J_{t+1}(y_t + z_t - D_t, \pi_{t+1})],$$

and π_T is defined for notational convenience. By introducing the variable $r_t = y_t + z_t - D_t$, we can write (2b) as

$$J_t(y_t, \pi_t) = \pi_t(D_t - y_t) + \min_{r_t \geq (y_t - D_t)_+} G_t(r_t, \pi_t), \tag{3}$$

where

$$G_t(r_t, \pi_t) = \pi_t r_t + \mathbb{E}[J_{t+1}(r_t, \pi_{t+1})].$$

It is clear that $\lim_{r_t \to \infty} G_t(r_t, \pi_t) = \infty$, since $J_{t+1}(r_t, \pi_{t+1}) = 0$ for all π_{t+1} when r_t is sufficiently large. We will prove shortly that $G_t(r_t, \pi_t)$ is convex in r_t. Thus, for a fixed π_t, $G_t(r_t, \pi_t)$ has a minimum $S_t(\pi_t)$ with respect to $r_t \geq 0$:

$$S_t(\pi_t) = \arg\min_{r_t \geq 0} G_t(r_t, \pi_t).$$

Then, in view of the constraint $r_t \geq y_t - D_t$, a minimizing r_t in (3) equals $S_t(\pi_t)$ if $y_t < S_t(\pi_t) + D_t$, and equals $y_t - D_t$ otherwise. Using the reverse transformation $z_t = r_t + D_t - y_t$, we see that the minimum in (2b) is attained at $z_t = S_t(\pi_t) + D_t - y_t$ if $y_t < S_t(\pi_t) + D_t$, and at $z_t = 0$ otherwise. We obtain an optimal policy in the following threshold form:

$$z_t^*(y_t, \pi_t) = \begin{cases} S_t(\pi_t) + D_t - y_t, & y_t < S_t(\pi_t) + D_t, \\ 0, & y_t \geq S_t(\pi_t) + D_t. \end{cases} \tag{4}$$

To show that $G_t(r_t, \pi_t)$ is convex in r_t, we proceed to show the convexity of the cost-to-go functions J_t inductively. Since $J_T(y_T, \pi_T)$ is the zero function, it is convex in y_T. Then $G_{T-1}(r_{T-1}, \pi_{T-1})$ is convex in r_{T-1}, and an optimal policy at time $T-1$ is given by

$$z_{T-1}^*(y_{T-1}, \pi_{T-1}) = \begin{cases} S_{T-1}(\pi_{T-1}) + D_{T-1} - y_{T-1}, \\ \qquad y_{T-1} < S_{T-1}(\pi_{T-1}) + D_{T-1}, \\ 0, \\ \qquad y_{T-1} \geq S_{T-1}(\pi_{T-1}) + D_{T-1}. \end{cases}$$

Furthermore, from (2b) we have

$$J_{T-1}(y_{T-1}, \pi_{T-1}) = \begin{cases} \pi_{T-1}(S_{T-1}(\pi_{T-1}) + D_{T-1} - y_{T-1}), \\ \qquad y_{T-1} < S_{T-1}(\pi_{T-1}) + D_{T-1}, \\ 0, \\ \qquad y_{T-1} \geq S_{T-1}(\pi_{T-1}) + D_{T-1}, \end{cases}$$

which is convex in y_{T-1}. Thus, given the convexity of J_T, we can establish the convexity of J_{T-1}. This argument can be repeated to show that for all $t = T-2, \ldots, 0$, if J_{t+1} is convex in y_{t+1}, then $G_t(r_t, \pi_t)$ is convex in r_t, and we have

$$J_t(y_t, \pi_t) = \begin{cases} \pi_t(S_t(\pi_t) + D_t - y_t) + \mathbb{E}[J_{t+1}(S_t(\pi_t), \pi_{t+1})], \\ \qquad y_t < S_t(\pi_t) + D_t, \\ \mathbb{E}[J_{t+1}(y_t - D_t, \pi_{t+1})], \\ \qquad y_t \geq S_t(\pi_t) + D_t, \end{cases}$$

which can be shown to be convex in y_t. Thus, the optimality of the threshold policy (4) is proved. ■

We now derive some important properties of the cost-to-go functions.

Proposition 2: $J_t(y_t, \pi_t)$ is decreasing in y_t for all t.

Proof: Since $J_T(y_T, \pi_T) = 0$, $J_T(y_T, \pi_T)$ is decreasing in y_T. Assume that $J_{t+1}(y_{t+1}, \pi_{t+1})$ is decreasing in y_{t+1}. Then for any y_t and $\hat{y}_t$ such that $\hat{y}_t > y_t$, we have $J_{t+1}(\hat{y}_t + z_t - D_t, \pi_{t+1}) \leq J_{t+1}(y_t + z_t - D_t, \pi_{t+1})$, so that

$Q_t(\hat{y}_t, \pi_t, z_t) \leq Q_t(y_t, \pi_t, z_t)$. Therefore,

$$\begin{aligned} J_t(\hat{y}_t, \pi_t) &= \min_{z_t \geq (D_t - \hat{y}_t)_+} Q_t(\hat{y}_t, \pi_t, z_t) \\ &\leq \min_{z_t \geq (D_t - y_t)_+} Q_t(\hat{y}_t, \pi_t, z_t) \\ &\leq \min_{z_t \geq (D_t - y_t)_+} Q_t(y_t, \pi_t, z_t) \\ &= J_t(y_t, \pi_t). \end{aligned}$$

The claim follows by induction. ■

Proposition 3: If $f_t(\cdot|\hat{\pi}_t)$ has first-order stochastic dominance over $f_t(\cdot|\pi_t)$ for all t and for all $\hat{\pi}_t$ and π_t such that $\hat{\pi}_t > \pi_t$, then $J_t(y_t, \pi_t)$ is increasing in π_t for all t.

Proof: Since $J_T(y_T, \pi_T) = 0$, $J_T(y_T, \pi_T)$ is increasing in π_T. Assume that $J_{t+1}(y_{t+1}, \pi_{t+1})$ is increasing in π_{t+1}. Then for any π_t and $\hat{\pi}_t$ such that $\hat{\pi}_t > \pi_t$, the first-order stochastic dominance gives

$$\begin{aligned} &\mathbb{E}[J_{t+1}(y_t + z_t - D_t, \pi_{t+1}) | y_t, \hat{\pi}_t, z_t] \\ \geq\ &\mathbb{E}[J_{t+1}(y_t + z_t - D_t, \pi_{t+1}) | y_t, \pi_t, z_t], \end{aligned}$$

which implies $Q_t(y_t, \hat{\pi}_t, z_t) \geq Q_t(y_t, \pi_t, z_t)$ and therefore $J_t(y_t, \hat{\pi}_t) \geq J_t(y_t, \pi_t)$. The claim follows by induction. ■

C. Elastic Demand Case

In this case, the user's demand is represented by a utility function $u_t(x_t)$ for each time t, which is concave, increasing and differentiable with $\lim_{x_t \to \infty} u_t'(x_t) = 0$. Since $x_t, y_t, z_t \geq 0$ for all t, the control constraint at time t given the state (y_t, π_t) is $x_t \geq 0, z_t \geq (x_t - y_t)_+$. The user's problem is to maximize the expected net utility:

$$\text{maximize } \mathbb{E}[\textstyle\sum_t (u_t(x_t) - \pi_t z_t)]. \tag{5}$$

Proposition 4: The optimal policy of problem (5) has a threshold form as shown in (8).

Proof: Define the (negative) cost-to-go functions as

$$J_T(y_T, \pi_T) = 0, \tag{6a}$$

$$J_t(y_t, \pi_t) = \max_{x_t \geq 0, z_t \geq (x_t - y_t)_+} Q_t(y_t, \pi_t, x_t, z_t), \tag{6b}$$

where

$$\begin{aligned} &Q_t(y_t, \pi_t, x_t, z_t) \\ &= u_t(x_t) - \pi_t z_t + \mathbb{E}[J_{t+1}(y_t + z_t - x_t, \pi_{t+1})]. \end{aligned}$$

By introducing the variable $r_t = y_t + z_t - x_t$, we can write (6b) as

$$\begin{aligned} &J_t(y_t, \pi_t) \\ &= \pi_t y_t + \max_{x_t \geq 0, r_t \geq (y_t - x_t)_+} F_t(x_t, r_t, \pi_t) \\ &= \pi_t y_t + \max_{x_t \geq 0, r_t \geq (y_t - x_t)_+} (G_t(x_t, \pi_t) + H_t(r_t, \pi_t)), \end{aligned} \tag{7}$$

where

$$\begin{aligned} F_t(x_t, r_t, \pi_t) &= G_t(x_t, \pi_t) + H_t(r_t, \pi_t), \\ G_t(x_t, \pi_t) &= u_t(x_t) - \pi_t x_t, \\ H_t(r_t, \pi_t) &= \mathbb{E}[J_{t+1}(r_t, \pi_{t+1})] - \pi_t r_t. \end{aligned}$$

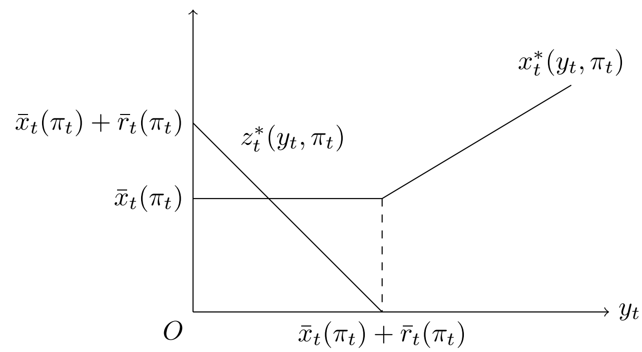

Fig. 1. The optimal policy $(x_t^*(y_t, \pi_t), z_t^*(y_t, \pi_t))$ of problem (5) as a function of y_t for a fixed π_t.

First, we show inductively the concavity of $J_t(y_t, \pi_t)$ in y_t. Since $J_T(y_T, \pi_T) = 0$, $J_T(y_T, \pi_T)$ is concave in y_T. Assume that $J_{t+1}(y_{t+1}, \pi_{t+1})$ is concave in y_{t+1}. Then $H_t(r_t, \pi_t)$ is concave in r_t. Also, $G_t(x_t, \pi_t)$ is concave in x_t. So $F_t(x_t, r_t, \pi_t)$ is concave in (x_t, r_t). Since partial maximization over a convex set preserves concavity, the second term on the right-hand side of (7) is concave in y_t. Therefore, $J_t(y_t, \pi_t)$ is concave in y_t.

Now we show that the optimal policy has a threshold form. Since $\lim_{x_t \to \infty} u_t'(x_t) = 0$, we have $\lim_{x_t \to \infty} G_t(x_t, \pi_t) = -\infty$ and $\lim_{r_t \to \infty} H_t(r_t, \pi_t) = -\infty$. Thus, for a fixed π_t, $G_t(x_t, \pi_t)$ has a maximum $\bar{x}_t(\pi_t)$ with respect to $x_t \geq 0$:

$$\bar{x}_t(\pi_t) = \arg\min_{x_t \geq 0} G_t(x_t, \pi_t),$$

and $H_t(r_t, \pi_t)$ has a maximum $\bar{r}_t(\pi_t)$ with respect to $r_t \geq 0$:

$$\bar{r}_t(\pi_t) = \arg\min_{r_t \geq 0} H_t(r_t, \pi_t).$$

Let $(\hat{x}_t(y_t, \pi_t), \hat{r}_t(y_t, \pi_t))$ be a solution to the following optimization problem:

$$\begin{aligned} \underset{x_t, r_t}{\text{maximize}} \quad & u_t(x_t) + \mathbb{E}[J_{t+1}(r_t, \pi_{t+1})] \\ \text{subject to} \quad & x_t + r_t = y_t, \\ & x_t, r_t \geq 0. \end{aligned}$$

Then, in view of the constraint $r_t \geq y_t - x_t$, a maximizing (x_t^*, r_t^*) in (7) equals $(\bar{x}_t(\pi_t), \bar{r}_t(\pi_t))$ if $y_t \leq \bar{x}_t(\pi_t) + \bar{r}_t(\pi_t)$; if $y_t > \bar{x}_t(\pi_t) + \bar{r}_t(\pi_t)$, there must be $y_t = x_t^* + r_t^*$, so that $(x_t^*, r_t^*) = (\hat{x}_t(y_t, \pi_t), \hat{r}_t(y_t, \pi_t))$. Using the reverse transformation $z_t = r_t + x_t - y_t$, we obtain an optimal policy in the following threshold form:

$$x_t^*(y_t, \pi_t) = \begin{cases} \bar{x}_t(\pi_t), & y_t \leq \bar{x}_t(\pi_t) + \bar{r}_t(\pi_t), \\ \hat{x}_t(y_t, \pi_t), & y_t > \bar{x}_t(\pi_t) + \bar{r}_t(\pi_t), \end{cases} \tag{8a}$$

$$z_t^*(y_t, \pi_t) = \begin{cases} \bar{x}_t(\pi_t) + \bar{r}_t(\pi_t) - y_t, & y_t \leq \bar{x}_t(\pi_t) + \bar{r}_t(\pi_t), \\ 0, & y_t > \bar{x}_t(\pi_t) + \bar{r}_t(\pi_t), \end{cases} \tag{8b}$$

as shown in Fig. 1. Note that the curve of $\hat{x}_t(y_t, \pi_t)$ on $y_t > \bar{x}_t(\pi_t) + \bar{r}_t(\pi_t)$ is plotted as linear for illustration purposes. In general, we can only claim that it is increasing in y_t. ■

III. System Problem: Stochastic Dynamic Pricing

Now we formulate the system problem, which involves multiple users served by a single distributor. The system objective is to maximize the expected social welfare. We show that there exists a price process such that when both the distributor and the users maximize their own payoffs, the resulting outcome is also socially optimal.

A. Model

Consider a finite horizon, where the time is indexed by $t = 0, 1, \dots, T-1$. There are N users, indexed by $n = 1, \dots, N$. The user's model is the same as that of the elastic demand case in the last section, except for the additional subscript n in the corresponding variables. Specifically, $x_{n,t}$ is the consumption of user n at time t, $y_{n,t}$ is the storage level of user n at the beginning of time t, and $z_{n,t}$ is the demand of user n at time t. The storage dynamics of user n is given by

$$y_{n,t+1} = y_{n,t} + z_{n,t} - x_{n,t}, \ \forall t.$$

There is a single distributor who sells $z_t \geq 0$ amount of power at each time t. We call z_t the supply at time t, which must match the demand:

$$\textstyle\sum_n z_{n,t} = z_t, \ \forall t.$$

The utility function of user n at time t is denoted by $u_{n,t}(x_{n,t})$, which is concave, increasing and differentiable with $\lim_{x_{n,t}\to\infty} u'_{n,t}(x_{n,t}) = 0$. Note that the utility functions are deterministic.

To model the uncertainty, we assume that the distributor has a stochastic cost function $c_t(z_t, W_t)$ for each time t, which is driven by an exogenous stochastic process $\{W_t, \forall t\}$ whose natural filtration is denoted by $\{\mathcal{F}_t, \forall t\}$. Moreover, the function $c_t(z_t, W_t)$ with respect to z_t is convex, increasing and differentiable with $\lim_{z_t\to\infty} c'_t(z_t, W_t) = \infty$.

For example, the cost function $c_t(z_t, W_t) = ((z_t - W_t)_+)^2$ can be interpreted in the following way. The distributor owns a wind farm, which generates W_t amount of power for free at time t. On top of that, the distributor can procure deterministic power at a quadratic cost.

Let $d = \{z_t, \forall t\}$ be the collection of the decision variables of the distributor, and $d_n = \{x_{n,t}, y_{n,t}, z_{n,t}, \forall t\}$ be that of each user n. The system problem is to maximize the expected social welfare:

$$\begin{aligned}
&\underset{d, d_n, \forall n}{\text{maximize}} && \mathbb{E}[\textstyle\sum_t(\sum_n u_{n,t}(x_{n,t}) - c_t(z_t, W_t))] && \text{(9a)}\\
&\text{subject to} && \textstyle\sum_n z_{n,t} = z_t, \ \forall t, && \text{(9b)}\\
& && y_{n,t+1} = y_{n,t} + z_{n,t} - x_{n,t}, \ \forall n, \ \forall t, && \text{(9c)}\\
& && x_{n,t}, y_{n,t}, z_{n,t}, z_t \geq 0, \ \forall n, \ \forall t, && \text{(9d)}\\
& && x_{n,t}, y_{n,t}, z_{n,t}, z_t \in \mathcal{F}_t, \ \forall n, \ \forall t. && \text{(9e)}
\end{aligned}$$

B. Dual Decomposition

We call the solution to the system problem (9) an efficient allocation. The goal is to show that there exists a price process according to which the efficient allocation can be achieved when each agent maximizes his own payoff. Such a price process is referred to as an optimal price process. To this end, we apply dual decomposition methods [17].

Define

$$\mathcal{D} = \{d \,|\, z_t \geq 0, z_t \in \mathcal{F}_t, \forall t\}$$

and

$$\mathcal{D}_n = \left\{ d_n \;\middle|\; \begin{array}{l} y_{n,t+1} = y_{n,t} + z_{n,t} - x_{n,t}, \forall t, \\ x_{n,t}, y_{n,t}, z_{n,t} \geq 0, \forall t, \\ x_{n,t}, y_{n,t}, z_{n,t} \in \mathcal{F}_t, \forall t \end{array} \right\}$$

for all n. Note that $\mathcal{D} \cup (\cup_n \mathcal{D}_n)$ is not the constraint set of the system problem (9). There is another coupling constraint (9b), which complicates the problem.

Associate the Lagrange multipliers π_t with (9b). Note that $\pi = \{\pi_t, \forall t\}$ is adapted to $\{\mathcal{F}_t, \forall t\}$. The (partial) Lagrangian is defined as

$$\begin{aligned}
&L(d, d_1, \dots, d_N, \pi)\\
&= \mathbb{E}[\textstyle\sum_t(\sum_n u_{n,t}(x_{n,t}) - c_t(z_t, W_t) + \pi_t(z_t - \sum_n z_{n,t}))].
\end{aligned}$$

The dual function is defined as

$$\begin{aligned}
g(\pi) &= \max_{d\in\mathcal{D}, d_n\in\mathcal{D}_n, \forall n} L(d, d_1, \dots, d_N, \pi)\\
&= h(\pi) + \textstyle\sum_n h_n(\pi),
\end{aligned}$$

where

$$h(\pi) = \max_{d\in\mathcal{D}} \mathbb{E}[\textstyle\sum_t(\pi_t z_t - c_t(z_t, W_t))], \quad (10)$$

$$h_n(\pi) = \max_{d_n\in\mathcal{D}_n} \mathbb{E}[\textstyle\sum_t(u_{n,t}(x_{n,t}) - \pi_t z_{n,t})]. \quad (11)$$

That is, the dual function $g(\pi)$ decouples into the distributor's problem (10) and user's problem (11) for all n. Let $d^*(\pi)$ be a solution to (10), and $d^*_n(\pi)$ be a solution to (11). The dual problem is then given by

$$\begin{aligned}
&\underset{\pi}{\text{minimize}} && h(\pi) + \textstyle\sum_n h_n(\pi) && \text{(12a)}\\
&\text{subject to} && \pi_t \in \mathcal{F}_t, \ \forall t. && \text{(12b)}
\end{aligned}$$

Since the system problem (9) is convex and satisfies the (refined) Slater's condition, strong duality holds. That is, when π^* is a dual optimal solution that solves (12), $\{d^*(\pi^*), d^*_n(\pi^*), \forall n\}$ is a primal optimal solution that solves (9). We state the result in the following.

Proposition 5: If π^* solves the dual problem (12), $d^*(\pi^*)$ solves the distributor's problem (10), and $d^*_n(\pi^*)$ solves the user's problem (11) for all n, then $\{d^*(\pi^*), d^*_n(\pi^*), \forall n\}$ solves the system problem (9).

The dual decomposition has the following economic interpretation. When the dual optimal solution is chosen as the price process, according to which each agent maximizes his own payoff, the resulting outcome is also socially optimal. That is, the dual optimal solution π^* is an optimal price process. We now give a more explicit form of π^*.

Proposition 6: Let $\{x^*_{n,t}, y^*_{n,t}, z^*_{n,t}, z^*_t, \forall n, \forall t\}$ be a primal optimal solution that solves (9). Then $\{\pi^*_t, \forall t\}$ with $\pi^*_t = c'_t(z^*_t, W_t)$ for all t is a dual optimal solution that solves (12).

Proof: Associate the Lagrange multipliers $\lambda_{n,t}$ with (9c). Since strong duality holds, the KKT conditions provide necessary and sufficient conditions for optimality [18]:

$$x_{n,t}(u'_{n,t}(x_{n,t}) - \lambda_{n,t}) = 0,\ \forall n,\ \forall t, \tag{13a}$$
$$u'_{n,t}(x_{n,t}) - \lambda_{n,t} \le 0,\ \forall n,\ \forall t, \tag{13b}$$
$$y_{n,t}(\lambda_{n,t} - \lambda_{n,t-1}) = 0,\ \forall n,\ \forall t, \tag{13c}$$
$$\lambda_{n,t} - \lambda_{n,t-1} \le 0,\ \forall n,\ \forall t, \tag{13d}$$
$$z_{n,t}(\lambda_{n,t} - \pi_t) = 0,\ \forall n,\ \forall t, \tag{13e}$$
$$\lambda_{n,t} - \pi_t \le 0,\ \forall n,\ \forall t, \tag{13f}$$
$$z_t(\pi_t - c'_t(z_t, W_t)) = 0,\ \forall t, \tag{13g}$$
$$\pi_t - c'_t(z_t, W_t) \le 0,\ \forall t, \tag{13h}$$
$$\textstyle\sum_n z_{n,t} - z_t = 0,\ \forall t, \tag{13i}$$
$$y_{n,t+1} - y_{n,t} - z_{n,t} + x_{n,t} = 0,\ \forall n,\ \forall t, \tag{13j}$$
$$x_{n,t}, y_{n,t}, z_{n,t}, z_t \ge 0,\ \forall n,\ \forall t, \tag{13k}$$
$$x_{n,t}, y_{n,t}, z_{n,t}, z_t, \pi_t, \lambda_{n,t}, \in \mathcal{F}_t,\ \forall n,\ \forall t. \tag{13l}$$

By (13h), $\pi_t^* \le c'_t(z_t^*, W_t)$ for all t. Suppose that $\pi_t^* < c'_t(z_t^*, W_t)$ for some t. It follows from (13g) that $z_t^* = 0$ and therefore $z_{n,t}^* = 0$ for all n by (13i) and (13k). It is easy to check that if we replace π_t^* by $c'_t(z_t^*, W_t)$ and keep the other variables unchanged, the KKT conditions (13) are still satisfied. Thus, $\{\pi_t^*, \forall t\}$ with $\pi_t^* = c'_t(z_t^*, W_t)$ for all t is a dual optimal solution. ■

C. Offline Distributed Algorithm

The dual decomposition (12) prompts an offline distributed algorithm to compute the optimal price process $\pi^* = \{\pi_t^*, \forall t\}$ that is adapted to $\{\mathcal{F}_t, \forall t\}$, the natural filtration for $\{W_t, \forall t\}$.

In general, W_t has an infinite support for all t; so does π_t^*. For computational tractability, assume that $\mathcal{F}_t$ is finite for all t, which can be done by quantization. Moreover, assume that $h(\pi)$ and $h_n(\pi)$'s are differentiable with bounded gradients. Then the following gradient method can be used:

$$\pi_t(k+1) = [\pi_t(k) - \alpha_k(z_t^*(\pi(k)) - \textstyle\sum_n z_{n,t}^*(\pi(k)))]_+,\ \forall t, \tag{14}$$

where k is the iteration index, and $\alpha_k > 0$ is a diminishing step size satisfying

$$\lim_{k\to\infty} \alpha_k = 0, \quad \textstyle\sum_{k=0}^{\infty} \alpha_k = \infty.$$

For example, we can choose $\alpha_k = 1/k$. We emphasize that $\pi_t(k)$ is a function of $\mathcal{F}_t$.

Under the above assumptions, the algorithm is guaranteed to converge [19]. That is, $\pi(k)$ will converge to the dual optimal solution π^* as $k \to \infty$.

Note that $x_{n,t}^*(\pi(k))$ does not appear in (14) since it has been internalized by the user. In other words, each user only needs to report his optimal demand $z_{n,t}^*(\pi(k))$ in each iteration.

To summarize, we present the following offline distributed algorithm, which computes the optimal price process at the beginning of the horizon:

Algorithm 1 Offline Distributed Algorithm for Stochastic Dynamic Pricing

Initialization: the system sets $k = 0$ and $\pi(0)$ a positive process that is adapted to $\{\mathcal{F}_t, t \in \mathcal{T}\}$, and then broadcasts $\pi(0)$ to the distributor and the users.

1) The distributor solves the distributor's problem (10) given $\pi(k)$ and reports the solution $\{z_t^*(\pi(k)), \forall t\}$ to the system;
2) Each user n solves the user's problem (11) given $\pi(k)$ and reports the solution $\{z_{n,t}^*(\pi(k)), \forall t\}$ to the system;
3) The system updates the price process according to (14) with $\alpha_k = 1/k$, broadcasts the new price process $\pi(k+1)$ to the agents, and sets $k \leftarrow k+1$ and go to step 1 (until satisfying termination criterion).

D. Investigating the Information Structure

We have conceptually provided a distributed algorithm to compute the optimal price process. For the algorithm to work, the embedded problems (10) and (11) have to be solved efficiently. Upon observing the constraint sets $\mathcal{D}$ and $\mathcal{D}_n$, we note that the distributer's problem (10) decouples in t, so that it can be solved efficiently. The user's problem (11), however, is a nontrivial stochastic program that is difficult to solve. The main issue is that the optimal price process $\{\pi_t^*, \forall t\}$ in general has a sophisticated structure in terms of its joint distribution, even if the underlying process $\{W_t, \forall t\}$ is simple enough. This suggests that it is important to understand the information structure of the stochastic processes involved in this model.

Recall that $\{W_t, \forall t\}$ is an exogenous stochastic process, with the natural filtration $\{\mathcal{F}_t, \forall t\}$. The efficient allocation $\{x_{n,t}^*, y_{n,t}^*, z_{n,t}^*, z_t^*, \forall n, \forall t\}$, as a primal optimal solution to the system problem (9), is adapted to $\{\mathcal{F}_t, \forall t\}$. The optimal price process $\{\pi_t^*, \forall t\}$, as a dual optimal solution, is also adapted to $\{\mathcal{F}_t, \forall t\}$.

Denote the natural filtration for $\{\pi_t^*, \forall t\}$ by $\{\mathcal{G}_t, \forall t\}$. One immediate observation is that $\{\mathcal{G}_t, \forall t\}$ is no more refined than $\{\mathcal{F}_t, \forall t\}$. That is, $\mathcal{G}_t \subset \mathcal{F}_t$ for all t, since $\mathcal{G}_t$ is the smallest σ-algebra such that $\pi_s^* \in \mathcal{G}_t$ for all $s \le t$. Also, $\mathcal{G}_t$ is not necessarily equal to $\mathcal{F}_t$. This is because $\{\mathcal{F}_t, \forall t\}$ is the natural filtration for a general stochastic process $\{W_t, \forall t\}$. It is possible that $c_t(z_t, W_t) = c_t(z_t, \hat{W}_t)$ for some W_t and $\hat{W}_t$ such that $W_t \ne \hat{W}_t$, in which case W_t and $\hat{W}_t$ induce the same optimal price π_t^* and therefore $\mathcal{G}_t \subsetneq \mathcal{F}_t$.

In the remainder of the subsection, we present a numerical example to demonstrate the fact that even if W_t's are independent, $\{\pi_t^*, \forall t\}$ is not necessarily a Markov process (with respect to its natural filtration $\{\mathcal{G}_t, \forall t\}$, and hence $\{\mathcal{F}_t, \forall t\}$ as well). This example also illustrates the advantage of stochastic dynamic pricing over deterministic dynamic pricing.

Example 1: Consider $n = 1$ and $T = 3$. That is, there is only one user (served by a single distributor), so that we will omit the subscript n in the corresponding variables. The user has a stationary utility function $u_t(x_t) = \log(x_t + 1)$ for

TABLE I
THE OPTIMAL PRICE PROCESS IN EXAMPLE 1

$W_0W_1W_2$	π_0^*	π_1^*	π_2^*
000	0.8058	0.7474	0.6553
001	0.8058	0.7474	1
010	0.8058	1	0.7308
011	0.8058	1	1
100	1	0.7873	0.6824
101	1	0.7873	1
110	1	1	0.7311
111	1	1	1

$t = 0, 1, 2$. The distributor has a cost function $c_t(z_t, W_t) = z_t^2 + W_t z_t$, where W_0, W_1, W_2 are independent Bernoulli random variables with success probability $p = 1/2$.

While this problem can be solved by dynamic programming, it is too involved to give an analytical representation of the optimal policy even for such a simple setting. Rather, since the number of sample paths is small (i.e., $2^3 = 8$), we treat the stochastic convex program (9) as a large-scale static convex program, the solution to which gives the optimal values of the decision variables for each sample path.

We list the optimal price process in Table I. In this example, the natural filtration generated by $\{\pi_0^*, \pi_1^*, \pi_2^*\}$ is the same as that for $\{W_0, W_1, W_2\}$. The result is intuitive. For instance, whenever $W_t = 1$, we have $\pi_t^* = 1$ so that $z_t^* = 0$. This is true because $u_t'(0) = c_t'(0, 1) = 1$, which means that it cannot be optimal to buy any power (to be consumed either at the current time or in the future) when the cost is high, whatever the current storage level is.

Clearly, we have $P(\pi_2^* \,|\, \pi_1^* = 1, \pi_0^* = 0.8058) \neq P(\pi_2^* \,|\, \pi_1^* = 1, \pi_0^* = 1)$. This shows that the optimal price process is not a Markov process, which is a major challenge from the computational point of view.

Furthermore, the expected social welfare under stochastic dynamic pricing is 0.2773. As a comparison, we consider deterministic dynamic pricing. One common approach is to use the expected cost at each time, i.e., $\mathbb{E}[c_t(z_t, W_t)] = z_t^2 + z_t/2$. Then the deterministic optimal prices are $\pi_0^* = \pi_1^* = \pi_2^* = 0.8508$, with the expected social welfare 0.1294. This is considerably smaller than that obtained by stochastic dynamic pricing, since deterministic dynamic pricing does not exploit the refined information as time advances. This illustrates the advantage of stochastic dynamic pricing over deterministic dynamic pricing.

IV. CONCLUSION

We formulate a stochastic dynamic pricing framework in the context of retail electricity markets, as an extension of the deterministic dynamic NUM problems. We first consider the user's problem and show that the optimal policy has a threshold form. We then proceed to the system problem, which involves multiple users served by a single distributor. We apply dual decomposition methods to show the existence of an optimal price process that incentives the agents to choose the socially optimal decisions. We develop a distributed algorithm and investigate the information structure of the involved stochastic processes via a numerical example, which also illustrates the advantage of stochastic dynamic pricing over deterministic dynamic pricing.

To illustrate the key idea, we have not taken into account additional constraints such as storage capacity limits and consumption quota. Those constraints can be easily incorporated into this model, and a similar argument follows.

The major challenge to implement stochastic dynamic pricing is the computational burden. In future work, we will develop solution methods for the embedded stochastic programs in the proposed algorithm. We will also address practical issues such as price volatility.

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