1 What we learnt this week

- Applications: Rubinstein bargaining
- Classes of games: one-shot simultaneous-move games
- Solution concepts: rationalizability

2 Problems

Problem 1: A simple Rubinstein bargaining game

Assume that if a player is indifferent between accepting and rejecting an offer, she accepts. What is the backwards-induction outcome of this game?

Problem 2: Bargaining strategies

Consider the Rubinstein bargaining game that we saw in class: Players A and B are bargaining over a pie of size 1. In period 0, A offers split \((x_0, y_0)\) to B, \(x_0 + y_0 \leq 1\). If B accepts, the game ends with split \((x_0, y_0)\). If B rejects, the game goes on to period 1. In period 1, B offers split \((x_1, y_1)\) to A. If A accepts, the game ends with split \((x_1, y_1)\). If A rejects, the game goes on to period 2. In period 2, A offers split \((x_2, y_2)\) to B. If B accepts, the game ends with split \((x_2, y_2)\). If B rejects, the game ends with payoffs \((0, 0)\).

What is the set of (pure) strategies available to each player?

Problem 3: Another alternating-offers bargaining game

Consider the following variation of the Rubinstein bargaining game that we saw in class (this follows problem MWG 9B7). Assume that a player’s utility from getting an amount \(x\) of the pie is equal to \(x\). Moreover, assume that the players do not discount payoffs (i.e., \(\delta_A = \delta_B = 1\)), but they incur a cost of \(c < 1\) when making an offer (only the player making the offer incurs this cost). Suppose that player A makes the offer in period 0 and the game can continue for a finite number of periods \(T\). If an agreement has not been reached after \(T\) periods, the game ends and the players each receive nothing. As in class, assume that if a player is indifferent between accepting and rejecting an offer, she accepts.

What is the backwards-induction outcome of this game? [Hint: Your answer should depend on whether \(T\) is odd or even.]
Problem 4: Solving a normal-form game by rationalizability

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Solve the game by rationalizability. Does rationalizability yield a unique prediction?

Problem 5: Rationalizable strategies

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<td>D</td>
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Show that in the game above, all actions are rationalizable.

Problem 6: Does the order matter?

Prove that the order of removal of non-rationalizable strategies does not affect the set of strategies that remain in the end, i.e., the set of rationalizable strategies.
3 Answers

Problem 1

Consider the last period of the game. Player 1 accepts player 2’s offer if she obtains a non-negative payoff. Then player 2 offers \( a = 0 \) and player 1 accepts. In the previous period, player 1 then makes an offer that leaves player 2 indifferent between accepting now and rejecting and obtaining \( 1 - 0 = 1 \) in the next period. That is, player 1 offers \( a = 1 \). Moving backwards, in the previous period, player 2 makes an offer that leaves player 2 indifferent between accepting now and rejecting and obtaining \( 2 - 1 = 1 \) in the next period. That is, player 2 offers \( a = 1 \). Finally, in the first period, player 1 makes an offer that leaves player 2 indifferent between accepting now and rejecting and obtaining \( 3 - 1 = 2 \) in the next period. That is, player 1 offers \( a = 2 \).

Hence, the backwards-induction outcome is that player 1 offers \( a = 2 \) in the first period and player 2 accepts.

Problem 2

A strategy for player A consists of an offer she makes in period 0; a decision to accept or reject player B’s offer in period 1 if player A’s initial offer is rejected; and an offer player A makes in period 2 if player B’s offer in period 1 is also rejected. Let

\[
\Omega = \{ (x, y) | x, y \in [0, 1], x + y \leq 1 \}
\]

Then, in period 0, A’s offer can be written as some \( o_0 \in \Omega \). In period 1, A’s decision to accept (\( a_1 \)) or reject (\( r_1 \)) B’s offer may depend on A’s initial offer as well as B’s offer in period 1. So we can write it as some function \( h_1 \) that maps from all possible offers by player 1 in period 0, 2’s rejection of that offer, and all possible offers by player 2 in period 1. That is, \( h_1 \in \{ h | h : \Omega \times \{ r_0 \} \times \Omega \rightarrow \{ a_1, r_1 \} \} \).

Similarly, in period 2, A’s offer can be written as a function \( j_2 \) of the history of play up to that period: \( j_2 \in \{ j | j : \Omega \times \{ r_0 \} \times \{ r_1 \} \rightarrow \Omega \} \). Hence, the set of strategies for player A, \( S_A \), is the set of all 3-tuples \( s_A = (o_0; h_1; j_2) \).

Similarly, for player B, the set of strategies, \( S_B \), is the set of all 3-tuples \( s_B = (f_0; k_1; l_2) \), where \( f_0 \in \{ f | f : \Omega \rightarrow \{ a_0, r_0 \} \} \), \( k_1 \in \{ k | k : \Omega \times \{ r_0 \} \rightarrow \Omega \} \), and \( l_2 \in \{ l | l : \Omega \times \{ r_0 \} \times \{ r_1 \} \rightarrow \Omega \} \).

Problem 3

Consider period \( T \). The offering player will offer \((1, 0)\) (where the first payoff is that of the offering player) and the offered player will accept. In period \( T - 1 \), the offering player will make an offer that leaves the other player indifferent between accepting now and rejecting and obtaining \( 1 - c \) in the next period. Thus, the offering player in period \( T - 1 \) will offer \((c, 1 - c)\). Similarly, in period \( T - 2 \), the offering player will make an offer that leaves the other player indifferent between accepting now and rejecting and obtaining \( c - c \) in the next period, so she will offer \((1, 0)\).

If \( T \) is odd, player A is the player who makes the offer in the last period. Then she will offer \((1, 0)\) in every period in which she makes the offer and accept an offer if she obtains at least \( 1 - c \). Player B will make an offer that gives \( 1 - c \) to player A and \( c \) to herself in every period in which she makes the offer, and accept an offer if she obtains a non-negative payoff. Hence, the backwards-induction outcome is that player A offers \((1, 0)\) in period 0 and B accepts.

If \( T \) is even, player B is the player who makes the offer in the last period. Then she will offer \((0, 1)\) in every period in which she makes the offer and accept an offer if she obtains at least
1 − c. Player A will make an offer that gives 1 − c to player B and c to herself in every period in which she makes the offer, and accept an offer if she obtains a non-negative payoff. Hence, the backwards-induction outcome is that player A offers (c, 1 − c) in period 0 and B accepts.

Problem 4

First, we can eliminate A4 since player 1 prefers A2 to A4 for any beliefs. Then, since player 2 knows that player 1 will not play A4, B2 dominates B3 for player 2 and we can thus eliminate B3. Next, since player 1 knows that player 2 will not play B3, now A3 is dominated by A1 and we can thus eliminate it. Note that there are beliefs for which A1 is optimal and for which A2 is optimal. Similarly, there are beliefs for which B1 is optimal and for which B2 is optimal. Hence, we cannot put further restrictions on the actions of the players, and we stop here. We cannot obtain a unique prediction by using rationalizability in this game.

Problem 5

We have to show that all actions are optimal for some beliefs about the other player’s actions (for beliefs that do not assign a positive probability to the other player playing a strategy that is never a best response).

For player 1, it is straightforward that playing U is optimal if she believes that player 2 will play L with higher probability than M, and playing D is optimal if she believes that player 2 will play M with higher probability than L. Both U and D are optimal for player 1 if she believes that player 2 will play L and M with equal probability.

For player 2, suppose she thinks that player 1 plays U with probability p and D with probability 1 − p. Then,

\[ U_2(L) = 3p \]
\[ U_2(M) = 3(1 − p) = 3 − 3p \]
\[ U_2(R) = 2p + 2(1 − p) = 2 \]

Hence, player 2 prefers to play L to M iff 3p > 3 − 3p ⇔ p > 1/2, and L to R iff 3p > 2 ⇔ p > 2/3. Similarly, player 2 prefers M to L iff p < 1/2, and M to R iff 3 − 3p > 2 ⇔ p < 1/3. Thus, all actions are rationalizable: if p > 2/3, L is optimal; if p < 1/3, M is optimal; and if p ∈ (1/3, 2/3), R is optimal.

Problem 6

Let the set of rationalizable strategies for player i after N rounds of deletion of non-rationalizable strategies be \( \Sigma_i^N \). Suppose \( s_i \) is not rationalizable given the strategies in \( \Sigma_{-i}^N \). Suppose further that \( s_i \) is not deleted in the N + 1 round. Then since \( s_i \) was not rationalizable given the strategies in \( \Sigma_{-i}^N \), it will clearly not be rationalizable given the strategies in \( \Sigma_{-i}^{N+1} \subset \Sigma_{-i}^N \). Thus, this strategy will be deleted in the next round.