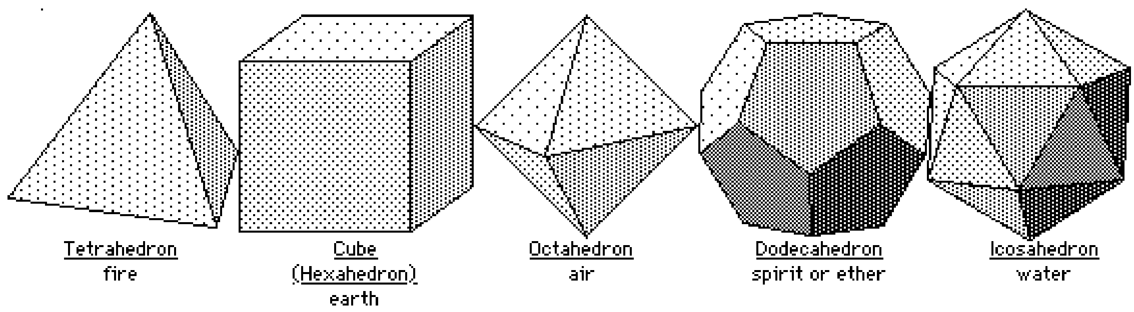
The Platonic Solids

William Wu
wwu@ocf.berkeley.edu

March 12 2004



The tetrahedron, the cube, the octahedron, the dodecahedron, and the icosahedron. From a first glance, one immediately notices that the Platonic Solids exhibit remarkable symmetry. They are the only convex polyhedra for which the same same regular polygon is used for each face, and the same number of faces meet at each vertex. Their symmetries are aesthetically pleasing, like those of stones cut by a jeweler. We can further enhance our appreciation of these solids by examining them under the lenses of group theory – the mathematical study of symmetry. This article will discuss the group symmetries of the Platonic solids using a variety of concepts, including rotations, reflections, permutations, matrix groups, duality, homomorphisms, and representations.

1 The Tetrahedron

1.1 Rotations

We will begin by studying the symmetries of the tetrahedron. If we first restrict ourselves to rotational symmetries, we ask, "In what ways can the tetrahedron be rotated such that the result appears identical to what we started with?" The answer is best elucidated with the

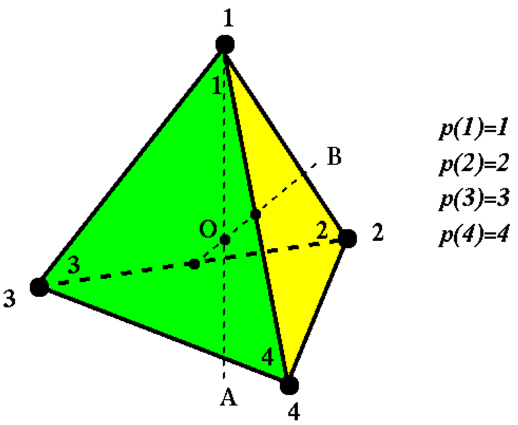


Figure 1: The Tetrahedron: Identity Permutation

aid of Figure 1. First consider the rotational axis $\overline{OA}$, which runs from the topmost vertex through the center of the base. Note that we can repeatedly rotate the tetrahedron about $\overline{OA}$ by $\frac{360}{3} = 120$ degrees to find two new symmetries. Since each face of the tetrahedron can be pierced with an axis in this fashion, we have found $4 \times 2 = 8$ new symmetries.

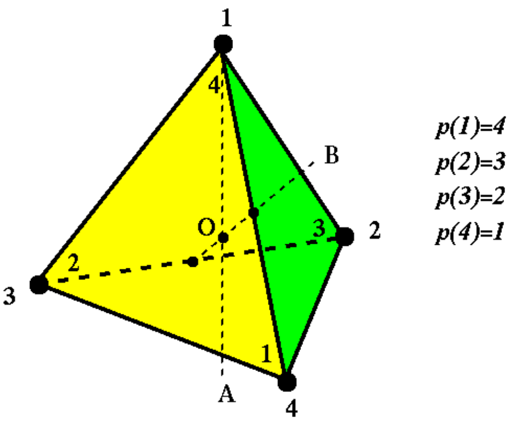


Figure 2: The Tetrahedron: Permutation (14)(23)

Now consider rotational axis $\overline{OB}$, which runs from the midpoint of one edge to the midpoint of the opposite edge. Rotating 180 degrees about this axis produces another symmetry. Figure 2 shows the tetrahedron's new orientation. Since there are three pairs of opposing edges whose midpoints can be pierced in this fashion, we have found 3 additional symmetries. Finally, we count the identity relation (no rotation) as a symmetry, yielding a total of 12 rotational symmetries along seven different axes of rotation.

1.2 Permutations

How can we be certain that all rotational symmetries have been found? We will argue this using permutations. We can uniquely label the tetrahedron's vertices as 1,2,3, and 4; in the figures above, the interior numbers represent these labels. By following how these numbers exchange their positions, we see that rotational symmetries can be put in a one-to-one correspondence with permutations of these four numbers. For example, the symmetry which maps Figure 1 to Figure 2 corresponds to the permutation (14)(23).

Now consider what generally happens when vertex v_1 moves to the position currently occupied by some other vertex v_2. This forces v_2 to move somewhere. Now, if v_2 swaps places with v_1, then the remaining vertices v_3 and v_4 must also swap places, since we are limiting ourselves to symmetries realizable by rigid motions. Thus this corresponds to a permutation of the form (12)(34). Alternatively, v_2 could take v_3's spot, which would in turn cause v_3 to take v_1's former spot, thus leaving v_4 fixed. This corresponds to a permutation of the form $(123) = (12)(13)$. In the last possible case where v_2 could take v_4's spot, the corresponding permutation is also of the form (12)(13). Thus we see that every non-trivial rotational symmetry corresponds to the product of two transpositions. This set of permutations, combined with the identity, forms the group of even permutations A_4. Since A_4 has order 12, all rotational symmetries must have been found, and the group of rotational symmetries is isomorphic to A_4.

1.3 Matrix Groups

Aside from permutations, we could also interpret our symmetries as a subgroup of $SO(3)$, which is the group of rotation matrices in $\mathbb{R}^3$. To compute these matrices, first fix coordinates for the vertices of the tetrahedron. Without loss of generality we can plant the tetrahedron's base against the xy-plane such that the base's center is at the origin, and one edge of the base is parallel to the x axis. Then geometric calculations yield the following coordinates:

$$\begin{aligned} c_1 &= (-1/2, -\sqrt{3}/6, -\sqrt{2/3}/4) \\ c_2 &= (1/2, -\sqrt{3}/6, -\sqrt{2/3}/4) \\ c_3 &= (0, 2\sqrt{3}/6, -\sqrt{2/3}/4) \\ c_4 &= (0, 0, 3\sqrt{2/3}/4) \end{aligned}$$

Now to compute the 3×3 rotation matrix M_p corresponding to permutation $\rho \in A_4$, first construct a 3×3 matrix A such that the i^{th} column contains the x,y,z coordinates of the i^{th} vertex. (Actually it does not matter which three distinct vertices we choose.) Now form a new matrix B_ρ such that the i^{th} column contains the x,y,z coordinates of the $(\rho(i))^{th}$ vertex. Then it follows that

$$M_\rho \cdot A = B_\rho$$

This is easy to see with a columnwise interpretation of matrix multiplication: M_p maps the i^{th} column of A to the i^{th} of B_ρ. Since any 3 vectors in $\{c_i\}_{i=1}^4$ are linearly independent, it follows that $\det(A) \neq 0$ and we can write $M_\rho = B_\rho \cdot A^{-1}$. Thus we can iteratively construct the matrix subgroup which is isomorphic to A_4.

1.4 Reflections

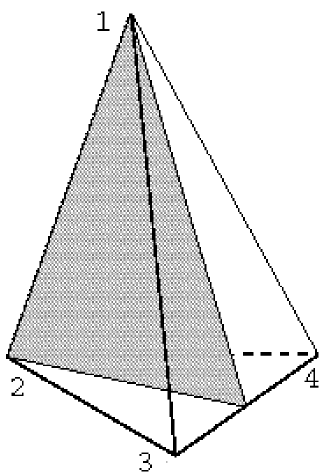


Figure 3: The Tetrahedron: A Reflection Plane (34)

How many symmetries can we find if we also allow reflections? We can find one new symmetry by transposing two vertices and leaving the other two alone. Figure 3 illustrated how this action is a reflection across the shaded plane. Let us denote this odd permutation by s. Note that $s^2 = e$. By composing each of the rotational symmetries with s, we can now find 12 new symmetries! Each composition must be unique, because if there existed two distinct elements $a_1, a_2 \in A_4$ such that $a_1 \cdot s = a_2 \cdot s$, then we could right-multiply each side by s to contradict the premise that a_1 and a_2 are distinct.

In general, the number of symmetries for a regular polyhedron must be upper-bounded by n!, where n is the number of vertices. This is because symmetries have the effect of changing the positions of vertices, and n! is the number of different ways n numbers can be permuted. In this case, n! = 24, so we have achieved the upper bound and there can be no more symmetries. Conclusively, the group of rotations and reflections of the tetrahedron is isomorphic to S_4, as well as the subgroup of $O(3)$ generated by adding reflections to the previously discussed subgroup of $SO(3)$.

In our survey of the tetrahedron we have seen three different ways of interpreting symmetries: rotations and reflections, permutations, and the matrix groups $O(3)$ and $SO(3)$. Each interpretation can also be applied to the other solids.

2 Cube and Octahedron

2.1 Rotations

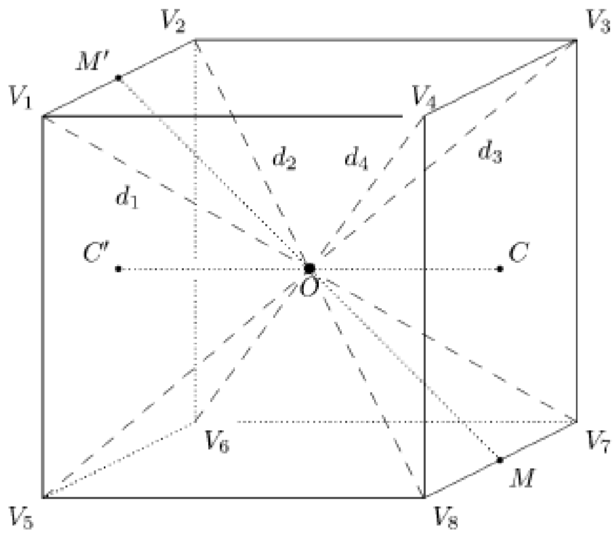


Figure 4: The Cube

What are the rotational symmetries of the cube? The most immediate symmetries are given by skewering the cube with a rotational axes that pierce through the centers of opposite faces. In Figure 4, $\overline{CC'}$ is an example of such an axis. We can repeatedly rotate the cube 90 degrees about this axis to procure three additional symmetries; since there are three pairs of opposite faces, we can find 9 symmetries of type. Eight more symmetries can be procured by skewering opposite corners (e.g. $\overline{V_1V_7}$) and repeatedly rotating by 120 degrees. And 6 more symmetries are given by skewering the midpoints of opposite edges and rotating by 180 degrees (e.g. $\overline{MM'}$) . Finally, we count the identity symmetry to tally 24 rotational symmetries.

More insight on the cube's symmetries can be gleaned by examining how rotations permute the cube's four principal diagonals d_1,d_2,d_3, and d_4. These are represented by dashed lines in Figure 4. Consider looking down at the face containing point C, and then rotating the cube 90 degrees clockwise about CC'. Some careful visualization will reveal that the diagonals are permuted to yield the 4-cycle (1234). Also visualize how a 180 degree rotation about MM' will yields the transposition (12). Since these two permutations are generators of S_4, it follows that our map ϕ from rotations to elements of S_4 is surjective. Finally, since the domain and range have the same size, it follows that the cube's rotational symmetries are isomorphic to S_4.

3 Reflections

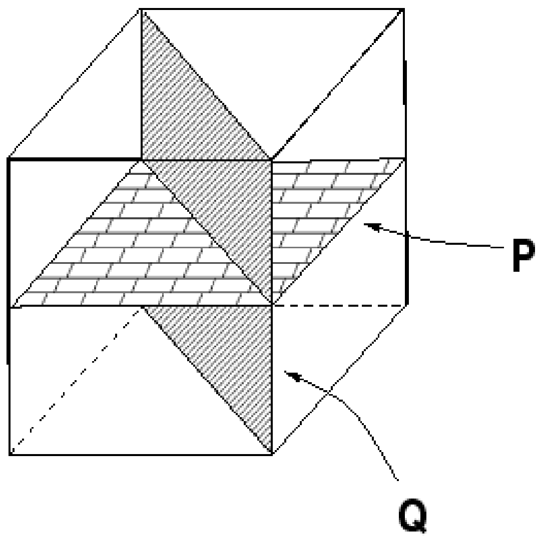


Figure 5: The Cube: Reflection Planes

The cube has reflectional symmetries across two different kinds of planes, illustrated in Figure 5 as P and Q. P slices the cube into two equally-sized rectangular prisms, while Q slices it into two equally-sized roofs. We can now find 24 new symmetries by choosing one reflection s, and composing the cube's rotational symmetries with s; the uniqueness is justified by repeating the argument given for tetrahedral reflective symmetries. These new symmetries can be thought of as the rotational symmetries of the original cube's mirror image.

We now claim that no more symmetries other than these 24 can be gleaned by adding reflections. To prove this, consider that the determinant is a homomorphism which maps:

$$\det : G \longmapsto \{\{1,-1\},\times\}$$

where G is the matrix subgroup of $O(3)$ consisting of reflective and rotational matrices for the cube, and $\{\{1,-1\},\times\}$ is the group isomorphic to $\mathbb{Z}_2$. This is a homomorphism since given any two matrices M_1 and M_2, the determinant of the matrix product equals the product of the matrix determinants. Now since the determinant of a rotation matrix is 1, the kernel of this homomorphism is known to be the group of 24 rotation matrices isomorphic to S_4. Since the homomorphism is surjective, it follows from the First Isomorphism Theorem that $G/K \cong \mathbb{Z}_2$. Namely, the correspondence $xK \longmapsto \det(x)$ is an isomorphism from the group of cosets $xK|x \in G$ to $\mathbb{Z}_2$. Now consider two distinct symmetries s_1 and s_2, perhaps one of which is of type P and the other of which is of type Q. Using the correspondence and the fact that reflective matrices have determinant -1, we have that $s_1K \longmapsto \phi(s_1) = -1$, and $s_2K \longmapsto \phi(s_2) = -1$. Since an isomorphism is a bijection, we must have that $s_1K = s_2K$. Thus there is only one nontrivial coset of the kernel, and reflections can only add 24 new symmetries.

4 Octahedral Dual

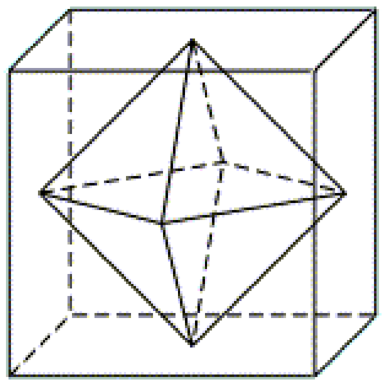


Figure 6: Duals: Octahedron Inscribed In A Cube.

Interestingly, the group structure of the octahedron is identical to that of the cube. This is not apparent at first, but can be deduced with a cunning observation. If we take a cube and place a dot in the center of each face, and then draw lines which connect these dots to their closest neighbors, we will have inscribed an octahedron inside a cube. Similarly, we can inscribe a cube inside an octahedron using the same procedure. The regularity of our Platonic solids then ensures that any rotation or reflection which is a symmetry for one solid must be a symmetry for its dual as well. The reader check this by verifying a few symmetries with the aid of Figure 6, which shows an octahedron inscribed in a cube.

It turns out that every Platonic Solid has a dual. The tetrahedron is its own dual.

5 Dodecahedron and Icosahedron

5.1 Rotations

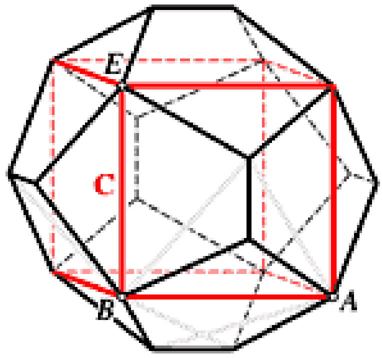


Figure 7: Inscription of Cube In Dodecahedron.

The rotational symmetries of the dodecahedron are isomorphic to A_5. To show this, we will consider how the dodecahedron's symmetries act on cubes which are inscribed in the dodecahedron. We inscribe these cubes in a way such that the volume of the cube is maximized within the constraints of the dodecahedral shell. There are 5 different cube orientations which satisfy this maximization, and in each case, every edge of the cube must align with the diagonal of a pentagonal face. (See Figure 7.) Now imagine inscribing all 5 cubes in the dodecahedron simultaneously, and numbering each one 1 through 5. Then the rotational symmetries of the dodecahedron correspond to permutations of the inscribed cubes. It turns out that by considering rotational axes which skewer through pairs of opposite vertices, one can show that the permutations generated by rotations are 3-cycles. It then follows that the rotational symmetries are isomorphic to A_5, since the 3-cycles generate A_5.

Note that a tetrahedron can be snugly inscribed in a cube by choosing any four non-adjacent vertices to be the vertices of the tetrahedron inside. Also, since there are two ways to choose quartets of non-adjacent vertices, it follows that there are two ways to inscribe a tetrahedron in a cube. From this idea we could consider inscribing tetrahedrons in the dodecahedron, as shown in Figure 8. Since there were 5 different orientations of inscribed cubes in the dodecahedron, it follows that there should be 10 different orientations of inscribed tetrahedra in the dodecahedron. Considering how dodecahedral rotations act on the compound structure formed by simultaneous inscription of 10 tetrahedra may offer a

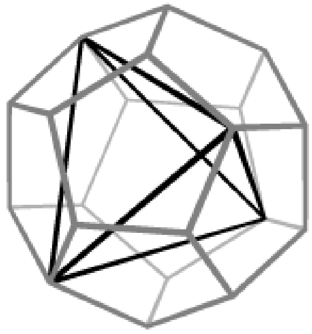


Figure 8: Inscription of Tetrahedron In Dodecahedron.

slightly different angle of attack in proving that the rotational symmetries are isomorphic to A_5. Another idea is to replace the cubes with their duals, the octahedra.

5.2 Icosahedral Duality

The dodecahedron and icosahedron are dual solids. See Figure 9.

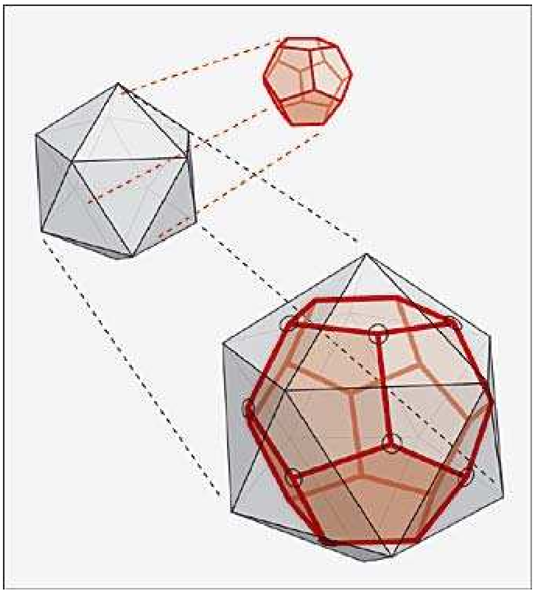


Figure 9: Duals: Dodecahedron Inscribed In An Octahedron.

6 Group Representations

Could representation theory reveal new information regarding the symmetries of the Platonic solids?

Let us consider S_4, the group of symmetries corresponding to both cubes and octahedra. The irreducible representations of S_4 are:

	irreducible representation	dimension
r1	identity	1
r2	permutation parity	1
r3	symmetries of the cube	3
r4	tensor product of permutation parity and cube symmetries	3
r5	homomorphism $\phi : S_4 \to S_3$	2

This is a complete list since it satisfies the dimensionality theorem: $|S_4| = 24 = 1^2 + 1^2 + 3^2 + 3^2 + 2^2$. The last two representations are the interesting ones which may offer new insight.

r4 can be thought of as S_4 acting a cube which has a signed bit attached to it. When the cube is hit with a permutation, the sign is also toggled if and only if that permutation is even. Thus it is similar to a direct product, but modulo an equivalence. Tensor products are difficult to visualize, so it is unclear to the student what this representation could suggest about the geometry of the solids.

r5 comes from the homomorphism which maps S_4 to S_3. Recall that the group S_4 contains the Klein subgroup V $= \{\epsilon, (12)(34), (13)(24), (14)(23)\}$. Now consider the group conjugation $H = \rho V \rho^{-1}$, where $\rho \in S_4$. Since V is normal, we know the sets H and V must contain the same elements; however, the order in which these elements appear can be different. Thus we can think of this group conjugation as a permutation of the three non-identity elements in V. So we have a surjective homomorphism $\phi : (\rho \in S_4) \longmapsto (\theta \in S_3)$. By applying the First Isomorphism Theorem, we conclude S_4/V is isomorphic to S_3. Since S_3 describes the symmetries of a triangle, S_3 is isomorphic to the symmetries of a subgroup of $O(2)$, which of course lies in $GL(2, \mathbb{R})$. Thus this representation has dimensionality 2.

What could this representation tell us visually about symmetries of the cube? If we determine which matrices in $O(3)$ correspond to the elements in the Klein kernel, we see that these correspond to the matrices that preserve pairs of opposite faces. This correspond to the symmetries of the cube in Figure 10, which had been modified by drawing one stripe on every face of the cube such that the stripes bisect the faces, and no two stripes meet. There are three face pairs, and the symmetries in the Klein kernel stabilize these pairs – each stripe must either stay still or swap places with the stripe on the opposite face. Now when we quotient out the kernel, we are in a sense ignoring the effects of symmetries in the

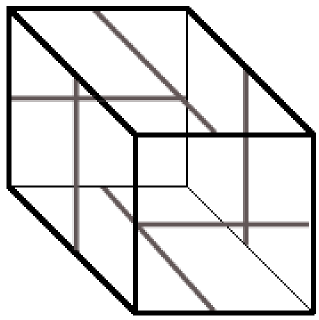


Figure 10: A Striped Cube.

kernel. Since there are three face pairs, the quotient group is S_3, which simply permutes these face pairs. Thus this group representation has revealed interesting symmetries about a striped cube, or equivalently, an octahedron in which each pair of opposite vertices has a unique color.

7 Further Studies

Given more time, the student might pursue several more ideas for further investigation of the Platonic solids. One idea is to find the representations of A_4 and A_5, and then try to determine what these representations mean geometrically. We could also examine how the conjugacy class partitionings of A_4, S_4, and A_5 could be better visualized by their corresponding partitionings of geometric symmetries in the solids. Another idea, inspired by wallpaper groups, is to investigate the possibility of using one Platonic solid as the building block for a three dimensional "wallpaper", or "spacefiller". While this is clearly possible for the cube, the question of whether the other solids could fill three dimensional space is not immediately obvious. We could also consider the possibility of forming a non-regular "spacepaper" using combinations of Platonic solids, minus those which *can* be used as regular "spacefillers". The existence of such "spacepapers" would strike an accord with the thoughts of Plato, who proposed that the Platonic solids were the fundamental building blocks of the universe.