Generalized ultraproducts for positive logic

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January 19, 2023

Ultraproducts as directed limits

Given:

- An index I
- A family of structures $(A_i)_{i \in I}$
- An ultrafilter D on I

consider the following direct system:

Directed set $(D, \supseteq)$

Structures $(\prod_{i \in X} A_i)_{X \in D}$

Homomorphisms $\pi_{XY} : \prod_X A_i \to \prod_Y A_i$ (natural projections)

Its direct limit is isomorphic to $\prod_D A_i$.

Problem

Replace:

- I *with a poset* $(I, \leq)$
- D *with a prime filter in the lattice* $\mathrm{Up}(I, \leq)$ *of up-sets of* $(I, \leq)$
- *the products with more general limits (in the category-theoretic sense)*

Prime products

Definition

1. Wellfounded forests are posets whose ppl down-sets are wellordered.
2. A family $(h_{ij} : M_i \to M_j \mid i \le j \in I)$ of homomorphisms is a "direct system" if $h_{jk} \circ h_{ij} = h_{ik}$
3. Given:
   - $(I, \le)$ a wellfounded forest
   - F a filter in $\mathrm{Up}(I, \le)$
   - $(h_{ij} : M_i \to M_j \mid i \le j \in I)$ a direct system of homs

$$M := \left\{ a \in \prod_{i \in I} M_i : \exists I' \in F \, \forall i \le j \in I' \; h_{ij}(a(i)) = a(j) \right\}.$$

$(a \equiv_F b \overset{\text{def}}{\iff} [\![a = b]\!] \in F)$ is a congruence on the reduct M to the algebraic sublanguage The filter product $\prod_F M_i$ is the quotient.
Interpret a relation R by $\prod_F R\bar{a} \iff [\![R\bar{a}]\!] \in F$.
If F is prime, we call $\prod_F M_i$ a prime product.

Examples

1. Every ultraproduct is a prime product ($(I, \leq) = (I, \Delta_I)$).
2. Direct limits on any wellorder I is a prime product ($F := \mathrm{Up}(I) \setminus \{\emptyset\}$).

Theorem

Ultraproducts of direct limits of structures along wellorders are filter products of those structures.

Problem

Does there exist a prime product that is not isomorphic to an ultraproduct of direct limits?

Positive logic I

Definition (Ben Yaakov and Poizat (2007))

1. A positive (existential) (or $\exists^+$) formula is a first-order formula built from atomic formulae (including $\bot$) by
2. A basic h-inductive formula is a first-order formula obtained by universally quantifying, finitely many times, a conditional between $\exists^+$ formulae. An h-inductive (or $\forall_2^+$) formula is a conjunction of basic h-inductive formulae.

Positive logic II

Example

1. The first-order theory of structures in an arbitrary quasivariety is equivalent to a strict universal Horn theory, which is h-inductive.
2. In the language of unital rings, the field axioms are h-inductive.

   Not the zero ring $0 = 1 \to \bot$

   No zero divisors $\forall x \, \forall y [xy = 0 \to x = 0 \vee y = 0]$

   Inverses of nonzero elements $\forall x [x = 0 \vee \exists y \, xy = 1]$

   etc.

Positive logic III

Counterparts of model theory à la A. Robinson can be developed for positive logic.

Definition

1. An immersion is a map preserving and reflecting all positive formulas
2. M is a positively existentially closed (pec) model
   $\iff$ All homomorphism from M to a model of T is an immersion

Theorem (Ben Yaakov and Poizat (2007))

For every model M of an h-inductive model T, there is a pec model M' of T and a homomorphism $M \to M'$.

etc.

Łoś

Theorem

Given:

- I *a wellfounded forest*
- $(h_{ij} : M_i \to M_j \mid i \leq j \in I)$ *a direct system*
- F *a prime filter of* $\mathrm{Up}(I)$,

and an arbitrary positive formula $\phi(\overline{x})$ *and a tuple* $\overline{a} \in \prod_F M_i$*:*

$$\prod_F M_i \models \phi(\overline{a}) \iff [\![\phi(\overline{a})]\!] \in F.$$

Wellfoundedness is necessary here.

Łoś-Suszko

Theorem

If ϕ is merely $\forall_2^+$, under the same assumptions

$$\prod_F M_i \models \phi(\bar{a}) \Longleftarrow [\![\phi(\bar{a})]\!] \in F.$$

Theorem

A class of structures is axiomatized by $\forall_2^+$ sentences if and only if K *is closed under ultraroots and prime products.*

Keisler-Shelah

A prime power is a prime product solely constructed from endomorphisms.

Theorem

TFAE:

1. *A and B have the same positive theory.*
2. *Some prime product of ultrapowers of A is isomorphic to some prime product of ultrapowers of B.*

If A and B are saturated, then the following condition is also equivalent: A and B have isomorphic prime powers.

Remark

1. The saturation requirement is necessary for the simpler statement.
2. In practice, the saturation requirement can often be dispensed with (especially with algebras).
3. The following conjecture follows from the GCH:

Conjecture

The following condition is also equivalent: Some prime power of an ultrapower of A is isomorphic to some prime power of an ultrapower of B.

Problem

Does an arbitrary structure have a universal ultrapower?
(M is universal $\iff$ M is $|M|^+$-universal)

Proof

With the saturation assumption:

- Since A is universal, there exists $h : A \to A$ that factors through an immersion from a pec model of the $\forall_2^+$ theory T of A.
- Let A_ω be the direct limit of the ω-sequence $A \xrightarrow{h} A \xrightarrow{h} \cdots$.
- A_ω is a pec model of T realizing enough types consisting of $\exists_1$ formulas.
- A_ω and B_ω are back-and-forth equivalent.
- Apply the original Keisler-Shelah theorem.

References

Ben Yaakov, I. and Poizat, B. (2007). Fondements de la Logique Positive. *The Journal of Symbolic Logic*, 72 (**4**), 1141–1162.

Poizat, M. and Yeshkeyev, A. (2018), Positive Jonsson Theories, *Logica Universalis*, 12, 101-127

Problems

1. When do we not need the saturation requirement?
2. Can we get rid of the GCH from the clearner version?
3. Are there more applications? (There is one in algebraic logic)

Counterparts

Example (Welfoundedness is necessary)

- Language: $\{P\}$ (unary predicate)
- Poset: $(\mathbb{Z}, \leq)$
- Structures: $A_i := (\mathbb{Z}, (-\infty, i])$
- Homomorphisms: Identities
- Prime filter: $F := \{\mathbb{Z}\}$

P is interepred by $\prod_F A_i$ as $\emptyset$.

Example (No Keisler-Shelah type theorem just with prime powers)

- Language: $\mathbb{Q} \cup \{\leq\}$
- Structures: $\mathbb{Q}^* := \mathbb{Q} \cup \{\infty\}$ (∞ is an upper bound of $\mathbb{Q}$)

They have the same positive theory. Neither $\mathbb{Q}$ or $\mathbb{Q}^*$ has nontrivial endomorphisms. Thus prime powers are reduced powers, so they preserve all Horn sentences. The existence (or absence) of the maximum is Horn.