RESEARCH STATEMENT

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Overview

My research centers around algebras motivated by various logics useful in describing topological spaces, games, automata, provability, and more.

First, I have studied presentation and duality theorems for such algebras. I have proved representation theorems using broader representational frameworks than previously studied in which strictly more algebras can be represented under milder assumptions, and showed that fundamental theorems of the standard representational frameworks extend to these more general frameworks.

Second, I study the canonical generic objects arising from any natural classes of algebras. There is a rich interplay between the combinatorics of the algebras that embed into the generic object and the topological dynamical properties of its automorphism group. This has allowed me to demonstrate properties of the automorphism group of the generic of a natural class of lattices including its non-amenability, simplicity, and failure of Roelcke precompactness.

Third, my ongoing research concerns the relation between *algebraic logic* and *positive logic*. By adapting a classic construction from algebraic logic to positive logic, I was able to exactly characterize complete axiomatizable classes of such structures in this setting in terms of explicit closure properties of the class of structures in question. In the future, I plan to explore more applications of representation and duality theorems for wider classes of algebraic-logical problems.

1. Research achievements

1.1. **Automorphism groups of countable ultrahomogeneous structures.**

A countable structure M is *ultrahomogeneous* if any isomorphism between finitely generated substructures thereof can be extended to an isomorphism of M. Paradigmatic examples include the total order of rationals and the (unique) countable atomless Boolean algebra. The automorphism group of such a countable structure can be regarded as a Polish (topological) group inheriting the topology of the Baire space $\mathbb{N}^{\mathbb{N}}$. Ultrahomogeneous structures have as many automorphisms as possible, and their properties manifest themselves in their automorphism groups, i.e., their symmetry.

I have studied the automorphism group of the countable ultrahomogeneous Heyting algebra L into which every finite Heyting algebra embeds, which exists and is unique up to isomorphism. Heyting algebras are mild generalizations of Boolean algebras motivated by intuitionistic logic, and L is a natural object of study in second-order intuitionistic propositional logic [4]. Motivations for studying the automorphism group of L include the fact that L is similar to better-known structures like the countable atomless Boolean algebra B while it is different from them in a number of different key points. For instance, L has infinitely many n-generated substructures up to isomorphism and thus is not ω-categorical, while better-studied ultrahomogeneous structures in natural finite signatures historically tend to be ω-categorical.

In my research [17], I first showed that $G := \mathrm{Aut}(L)$ is not *Roelcke precompact*. (A topological group is Roelcke precompact if the completion of the group with respect to the meet of its left and right uniformities, respectively.) Since the direct limit of the automorphism groups of any ω-categorical countable ultrahomogeneous structures is Roelcke precompact, this result amounts to G being strictly different from better-known topological groups arising as the automorphisms groups of countable ω-categorical structures. In addition, I also showed that G is not amenable and that G is simple as an abstract group. The former follows from combinatorics of finite Heyting algebras, and the latter from the Craig interpolation theorem for intuitionistic propositional logic. Furthermore, in a different article [18], I showed the *small index property* of G, which amounts to saying that the topology of G is uniquely determined by its abstract group structure and the property that G is a closed subgroup of $\mathrm{Sym}(\mathbb{N})$. The method was based on the representation of L by a certain partial order $\leq$ on the Cantor set 2^ω such that the group of homeomorphisms of 2^ω preserving $\leq$ is exactly L.

1.2. **Duality theory for algebras arising from logic.** Most often, we axiomatically define a class $\mathcal{V}$ of algebras, and paradigmatic examples X^+ in $\mathcal{V}$ are constructed in a specific manner from some mathematical object X in some other class $\mathcal{F}$. For instance, with the class of Boolean algebras, X is a pure set, and X^+ is the powerset of X; with the class of groups, X is again a pure set, and $X^+ = \mathrm{Sym}(X)$. One is then interested in obtaining an arbitrary member $A \in \mathcal{V}$ as a subalgebra of $(\mathrm{Cst}\, A)^+$ where $\mathrm{Cst}\, A$, the *canonical structure* of A, is in $\mathcal{F}$, or encoding which subalgebra of $(\mathrm{Cst}\, A)^+$ it is using an appropriate topology on $\mathrm{Cst}\, A$. In the most basic setting of this kind that is relevant to the following achievements of mine, $\mathcal{V}$ is the class of *Boolean algebras with operators*—Boolean algebras expanded with finitely additive operations—and $\mathcal{F}$ is the class of sets with a binary relation on it. The following is a brief summary of classical results in this setting:

- **Definability:** One can use an equation E for $\mathcal{V}$ to define a subclass $\mathcal{K}$ of $\mathcal{F}$: $X \in F$ belongs to $\mathcal{K}$ if and only if X^+ satisfies E. In *correspondence theory*, one compares this notion of definability with other means of defining a subclass of $\mathcal{F}$ such as definability in first-order logic. One classical result in this field is the *Sahlqvist Correspondence Theorem*, which gives a syntactic sufficient condition on E under which the corresponding subclass of F is also definable in first-order logic. Additionally, one may be interested in characterizing this notion of definability in a way that does not mention the syntax explicitly, just as first-order definability of classes can be characterized in terms of closure properties. The Goldblatt-Thomason theorem [7] does this in terms of closure properties under constructions characteristic of the class $\mathcal{F}$.
- **Completeness:** One important aspect of studying algebras motivated by logic is to examine subclasses of $\mathcal{V}$ that can be defined by a set of equations, i.e., *varieties*. An important property of a variety is *completeness*. A variety $\mathcal{V}'$ is complete if the least variety containing $\{X^+ \mid X \in \mathcal{F}, X^+ \in \mathcal{V}'\}$ is $\mathcal{V}'$ itself. Complete varieties $\mathcal{V}'$ are easier to study in the sense that the study of $\mathcal{V}'$ reduces to its members of the form X^+, which are easier to analyze. One important sufficient condition for a variety to be complete is *canonicity*. A variety $\mathcal{V}'$ is canonical if for every $A \in \mathcal{V}'$ the algebra $(\mathrm{Cst}\, A)^+$ is also in $\mathcal{V}'$. Every canonical variety is complete; in fact, most completeness results have involved canonicity. Yet again, there is a classical result in this field, proved by Fine [5], concerning the relationship between first-order definability and canonicity.

In my article [15], I studied correspondence theory in the setting of possibility semantics [8]. In possibility semantics, $\mathcal{F}$ is the class of certain structures with a binary relation and a partial order, which are used to represent complete and completely additive Boolean algebras with operators as structures of the form X^+. This is more general than the aforementioned framework, where structures of the form X^+ are necessarily atomic. I proved that the same syntactic condition on equations E as in Sahlqvist's result guarantees that the class $\{X \in \mathcal{F} \mid X^+ \text{ satisfies } E\}$ is also definable in first-order logic.

In the article [16], I looked at the problem of non-syntactic characterization of definability and canonicity in the setting where $\mathcal{V}$ is the class of *monotonic Boolean algebra expansions*, Boolean algebras with additional operations that are merely monotonic. In this setting, $\mathcal{F}$ is a class of certain higher-order structures. In proving a result like Fine's, one has to make sense of "first-order definability" within such a class. I identified a logic [3] for $\mathcal{F}$ that shares important properties with first-order logic and used it to establish a similar relationship between canonicity in $\mathcal{V}$ and definability of subclasses of $\mathcal{F}$ in that logic. In the proof, I used the duality between types and definable subsets in model theory as guidance. This solved a problem asked in [11]. By using the same method, I proved a non-syntactic characterization of definability of subclasses of $\mathcal{F}$ in the same style as the Goldblatt-Thomason theorem.

In [12], I extended the work by Bezhanishvili and Holliday [2] on choice-free duality for Boolean algebras, and I developed a choice-free duality theory for ortholattices. I established the dual equivalence between the category of ortholattices and the category of certain spectral spaces and certain spectral maps without using any choice principles. This is a choice-free analogue of the work by Goldblatt [6], which I examined and compared with my approach; in particular, I characterized the duals of ortholattices under Goldblatt's representation theorem and I extended his result to the dual equivalence between the appropriate categories. I also developed a dictionary of how algebraic notions and constructions can be translated relationally and topologically in terms of their dual under the choice-free duality.

2. Future research plans

Homogeneous structures and topology. I plan to continue the study of the correspondence between the combinatorics of substructures of a countable ultrahomogeneous structure and the property of its automorphism group. In an ongoing work, I attempt to show the existence of an *extremely amenable* closed subgroup $G^* \leq G$ such that the quotient G/G^* has a compact completion. A topological group is *extremely amenable* if every continuous action onto a compactum has a fixed point. The methodology is the Ramsey theory of finite Heyting algebras and the so-called Kechris-Pestov-Todorčević correspondence [10]. Also I intend to study other properties of Polish groups, such as the Bergman property, in this setting.

Other research trends regarding homogeneous structures and topology include the construction of various continua by first constructing a binary relation R on the Cantor set 2^ω via the categorical dual of the Fraïssé construction of ultrahomogeneous structures and then taking the quotient of 2^ω by R. For instance, the pseudo-arc can be constructed in this way and its homogeneity in the topological sense follows easily from the construction [9]. The aforementioned partial order $\leq$ used in the study of the generic Heyting algebra L is actually an example of a binary relation constructed in this manner. In most work in this area, binary relations giving rise continua are made up artificially, but $\leq$ arises quite naturally. I aim to investigate the quotient of 2^ω with respect to the natural binary relation.

Positive logic. I also plan to do research that is motivated by algebraic logic but is relevant in more modern model theory. Generally speaking, in algebraic logic, the primary object of study is varieties, or classes of algebras defined by universally quantified equations. Recently, however, it was shown [13] that behaviors of non-universal first-order formulas over logically motivated algebras can affect certain properties of the corresponding logicrelating concerning derivability and admissibility of rules. Such formulas fall within the realm of positive logic [14]. In an ongoing work, I investigate the counterparts of ultraproducts in positive logic, whose construction was inspired by the topological representation of Heyting algebras, and in particular prove the analog of the Keisler-Shelah theorem. Stability theory for positive logic [1] has an interesting feature allowing certain ordered structures to be stable. In light of this, I aim to find more applications of positive logic in logically motivated algebras, which are almost always ordered.

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