Further Examples of Epsilon-Delta Proof

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The limit is formally defined as follows:

$$\lim_{x \to a} f(x) = L$$

if for every number $\epsilon > 0$ there is a corresponding number $\delta > 0$ such that

$$0 < |x - a| < \delta \quad \Longrightarrow \quad |f(x) - L| < \epsilon$$

Intuitively, this means that for any ϵ, you can find a δ such that $|f(x)-L| < \epsilon$.

To do the formal $\epsilon - \delta$ proof, we will first take ϵ as given, and substitute into the $|f(x) - L| < \epsilon$ part of the definition. Then we will try to manipulate this expression into the form $|x - a| <$ something. We will then let δ be this "something" and then using that δ, prove that the $\epsilon - \delta$ condition holds. Some examples should make this clear.

1. Prove:

$$\lim_{x \to 4} x = 4$$

We must first determine what a and L are. In this case, $a = 4$ (the value the variable is approaching), and $L = 4$ (the final value of the limit). The function is $f(x) = x$, since that is what we are taking the limit of.

Following the procedure outlined above, we will first take epsilon, as given, and substitute into $|f(x) - L| < \epsilon$ part of the expression:

$$|f(x) - L| < \epsilon \quad \Longrightarrow \quad |x - 4| < \epsilon$$

In this case we are lucky, because the expression has naturally simplified down to the form $|x - a| < \delta$! Therefore, since we know from the above that $|x - 4| < \epsilon$, we can let $\delta = \epsilon$, and we know that $|x - 4| < \delta$. This last point is very important.

We can now finish the proof:

Given ϵ, let $\delta = \epsilon$. Then:

$$\begin{aligned} |x - a| < \delta &\Longrightarrow |x - 4| < \epsilon \\ &\Longrightarrow |f(x) - L| < \epsilon \end{aligned}$$

This completes the proof.

2. Prove:

$$\lim_{x \to 1} 3x - 1 = 2$$

In this problem, $a = 1$, $L = 2$, $f(x) = 3x - 1$. We will proceed as we did before, by substituting into $|f(x) - L| < \epsilon$.

$$\begin{aligned} |f(x) - L| < \epsilon &\implies |(3x - 1) - 2| < \epsilon \\ &\implies |3x - 3| < \epsilon \\ &\implies 3|x - 1| < \epsilon \\ &\implies |x - 1| < \epsilon/3 \end{aligned}$$

Once again, we have simplified the expression down to the form $|x-a| < \delta$. In this case, we can let $\delta = \epsilon/3$, and we can write up the proof: Given ϵ, let $\delta = \epsilon/3$. Then:

$$\begin{aligned} |x - a| < \delta &\implies |x - 1| < \epsilon/3 \\ &\implies 3|x - 1| < \epsilon \\ &\implies |3x - 3| < \epsilon \\ &\implies |(3x - 1) - 2| < \epsilon \\ &\implies |f(x) - L| < \epsilon \end{aligned}$$

This completes the proof.

3. Prove:

$$\lim_{x \to \infty} \sqrt{x} = \infty$$

In this problem, we have $a = \infty$ and $L = \infty$. If we try to apply the proof directly, we will end up $|f(x) - \infty| < \epsilon$, which produces a meaningless result, since, anything minus ∞ is ∞. Therefore, we need to modify or definition of limit slightly for infinity problems.

Let us first consider what it means for the limit to be equal to infinity. When the limit equals infinity, then that means that $f(x)$ increases without bound. In other words, for any positive integer M, there is a δ such that within $0 < |x-a| < \delta$, $f(x) > M$ (mathematically, this is saying that as you approach the point a, $f(x)$ becomes larger than any finite value.

Now let us consider the other side: $x \to a$. If we directly substitute $a = \infty$, following the same arguments as the last paragraph, we end up with the same meaningless values. Therefore, we will also use a similar argument to replace $0 < |x - a| < \delta$ with $x > N$, where N is any positive integer.

Therefore, our modified definition of $\epsilon - \delta$ for this problem will be:

$$\lim_{x \to \infty} f(x) = \infty$$

if for every positive integer M there is a corresponding positive integer N such that

$$x > N \implies f(x) > M$$

* You should work out for yourself, what the precise definitions of limit would be if only one of x or f(x) is infinity, or if one of these is negative infinity, etc.

Now we can proceed as before:

$$\begin{aligned} f(x) > M &\implies \sqrt{x} > M \\ &\implies x > M^2 \end{aligned}$$

Here, we have matched the form $x > N$, which is our goal in this case. Now we can write the proof: Given a positive integer M, let $N = M^2$. Certainly N is a positive integer. Then:

$$\begin{aligned} x > N &\implies x > M^2 \\ &\implies \sqrt{x} > M \\ &\implies f(x) > M \end{aligned}$$

This completes the proof.

4. Prove:

$$\lim_{x \to 2} x^2 + x - 2 = 4$$

We start the same way we always do. By now this part should be automatic.

$$\begin{aligned} |f(x) - L| < \epsilon &\implies |(x^2 + x - 2) - 4| < \epsilon \\ &\implies |(x^2 + x - 6)| < \epsilon \\ &\implies |(x + 3)(x - 2)| < \epsilon \\ &\implies |x + 3||x - 2| < \epsilon \\ &\implies |x - 2| < \frac{\epsilon}{|x + 3|} \end{aligned}$$

Once again we have the left side in the form $|x - a|$. Now we can let δ equal $\frac{\epsilon}{|x+3|}$. But what do we do about the $|x + 3|$. In general δ must be in terms of ϵ only, without any extra variables.

So how we can remove this $x + 3$ term?

First we need to simplify the problem a little bit. Since the concept of limit only applies when x is close to a, we will first restrict x so that it is at most 1 away from a, or, mathematically, $|x - a| < 1$ (in our case $|x - 2| < 1$. Then, this means, $1 < x < 3$, or $4 < x + 3 < 6$.

Now consider the original inequality

$$|x - 2| < \frac{\epsilon}{|x + 3|}$$

Notice that the right hand side is at the minimum when $x+3$ is at its *maximum*. Since the maximum of $x+3$ is 6, we know that

$$|x-2| < \frac{\epsilon}{|x+3|} < \frac{\epsilon}{6}$$

Since we now have two restriction,

$$|x-2| < 1 \qquad \text{and} \qquad |x-2| < \frac{\epsilon}{6}$$

we let $\delta = min\{1, \frac{\epsilon}{6}\}$, the smaller of these two values, which guarantees that it will satisfy both inequalities.

Finally, after all this, we can write up the proof:

Given ϵ, let $\delta = min\{1, \frac{\epsilon}{6}\}$.

Suppose $\delta = 1$. Since $1 < \frac{\epsilon}{6}$, we know $\epsilon > 6$. Then:

$$\begin{aligned}
|x-a| < \delta &\implies |x-2| < 1 \\
&\implies |x-2||x+3| < |x+3| \\
&\implies |x^2+x-6| < |x+3|
\end{aligned}$$

We know that $4 < x+3 < 6$, using the same restriction we worked out earlier. We also know that $1 < \frac{\epsilon}{6}$ (because δ is the minimum of those two values). Therefore $\epsilon > 6$. Putting these together, we have

$$\begin{aligned}
|x^2+x-6| < |x+3| &\implies |x^2+x-6| < |x+3| < 6 < \epsilon \\
&\implies |(x^2+x-2)-4| < \epsilon \\
&\implies |f(x)-L| < \epsilon
\end{aligned}$$

This completes the case $\delta = 1$.

Now suppose $\delta = \frac{\epsilon}{6}$. Then:

$$|x-a| < \delta \implies |x-2| < \frac{\epsilon}{6}$$

Once again we know that $4 < x+3 < 6$, therefore

$$\begin{aligned}
|x-2| < \frac{\epsilon}{6} &\implies |x-2|(x+3) < 6\frac{\dot{\epsilon}}{6} \\
&\implies |x^2+x-6| < \epsilon \\
&\implies |(x^2+x-2)-4| < \epsilon \\
&\implies |f(x)-L| < \epsilon
\end{aligned}$$

Finally, this completes the proof.